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Note

On Markov's and Bernstein's Inequalities in the Unit Ball in $\mathbf{R}^k$

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Simple proofs and refinements of polynomial inequalities in the unit ball in $\mathbf{R}^k$ by O. D. Kellogg are given.

INTRODUCTION

This note deals with a generalization of Markov's inequality and Bernstein's inequality to several variables. Markov's inequality in one variable states that $\|P'\|_{[-1,1]} \leqslant n^2 \|P\|_{[-1,1]}$ for every polynomial P of degree less than or equal to n, where $[-1, 1]$ denotes the closed interval from -1 to 1 and the norm is the maximum norm. Using this inequality in different directions, one easily obtains a generalization to the closed unit ball $B(0, 1)$ in the Euclidian space $\mathbf{R}^k$. The result one gets in this way is that $\|\nabla P\|_{B(0,1)} \leqslant kn^2 \|P\|_{B(0,1)}$ for all polynomials P in k variables of total degree $\leqslant n$, where ∇ denotes the gradient, see, e.g., Coatmelec [1]; however, one does not expect this to be the best possible result. In fact, already O. D. Kellogg showed in [2] that one has $\|\nabla P\|_{B(0,1)} \leqslant n^2 \|P\|_{B(0,1)}$ in the k-dimensional case, too, and he gave a similar result as a generalization of Bernstein's inequality. See also Wilhelmsen [4].

In this note we give a very simple proof of Kellogg's inequalities. We also give refined versions, using a more elaborate method, which is close to the method used in [2].

1. BERNSTEIN'S AND MARKOV'S INEQUALITIES

Let $k \geqslant 1$ and denote by P_n the set of all polynomials in k variables of total degree $\leqslant n$. The following results are given in [2].

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THEOREM 1. *Let $P \in P_n$. Then, for $|\mathbf{x}| < 1$,*

$$|\nabla P(\mathbf{x})| \leqslant \frac{n}{\sqrt{1-|\mathbf{x}|^2}} \|P\|_{B(0,1)}.$$

THEOREM 2. *Let $P \in P_n$. Then*

$$\|\nabla P\|_{B(0,1)} \leqslant n^2 \|P\|_{B(0,1)}.$$

We note first that it is enough to prove the theorems for $k=2$. The proof of the general case then follows by using the result for $k=2$ in the two-dimensional subspace of $\mathbf{R}^k$ containing the vector $\mathbf{x}$ and the gradient vector of P at $\mathbf{x}$ (if these vectors are not parallel; in case they are, one can use the one-dimensional inequality).

Proof of Theorem 1. Let $\mathbf{x} = (x, y)$ be a point in $B(0, 1)$, let $\mathbf{v}$ be a unit vector and L the chord in $B(0, 1)$ through $\mathbf{x}$ in the direction $\mathbf{v}$. Denote by $2d$ the length of L, let $\mathbf{p}$ be the mid point of L, and put $Q(t) = P(\mathbf{p} + t\mathbf{v})$, $t \in \mathbf{R}$. Then Q is a polynomial in one variable of degree $\leqslant n$, and Bernstein's inequality for algebraic polynomials gives $|Q'(t)| \leqslant (n/\sqrt{d^2 - t^2}) \|Q\|_{[-d,d]}$, $|t| < d$. But for t corresponding to $\mathbf{x}$ holds $d^2 - t^2 = 1 - |\mathbf{x}|^2$ according to the theorem of intersecting chords, so we get

$$|P_{\mathbf{v}}(\mathbf{x})| \leqslant \frac{n}{\sqrt{1-|\mathbf{x}|^2}} \|P\|_L \leqslant \frac{n}{\sqrt{1-|\mathbf{x}|^2}} \|P\|_{B(0,1)} \tag{1}$$

which gives the estimate in Theorem 1; here $P_{\mathbf{v}}$ denotes the directional derivative.

In the proof of Theorem 2, which is essentially the same as in [2], and in the proof of Theorem 2′ below, we need the following lemma which is a well-known tool in the proof of Markov's inequality, see, e.g., [3, p. 139].

LEMMA 1. *Let P be a polynomial of degree $\leqslant n-1$ in one variable. Assume that $|P(x)| \leqslant 1/\sqrt{d^2 - x^2}$, $x \in (-d, d)$. Then $|P(x)| \leqslant n/d$, $x \in [-d, d]$.*

Proof of Theorem 2. Let $\mathbf{v}$ be a unit vector, and consider a diameter D. The directional derivative $P_{\mathbf{v}}$ is a polynomial of degree $\leqslant n-1$, and on D we may view it as a polynomial in one variable by setting $Q(t) = P_{\mathbf{v}}(t\mathbf{v}_1)$, where $\mathbf{v}_1$ is a unit vector in the direction of D. By (1) we have $|Q(t)| \leqslant (n/\sqrt{1-t^2}) \|P\|_{B(0,1)}$. Thus Lemma 1, used with $d = 1$, gives the estimate $\|P_{\mathbf{v}}\|_D \leqslant n^2 \|P\|_{B(0,1)}$ and the theorem follows.

2. Refined Inequalities

Our first result is a refinement of Bernstein's inequality. For a still more precise estimate, see (2) below.

Theorem 1′. *Let $P \in P_n$ and $|\mathbf{x}| < 1$, let $\mathbf{v}$ be a unit vector, and denote by $2d$ the length of the chord L through $\mathbf{x}$ in the direction $\mathbf{v}$. Then*

$$|P_{\mathbf{v}}(\mathbf{x})| \leqslant \frac{nd}{\sqrt{1-|\mathbf{x}|^2}} \|P\|_{B(0,1)}.$$

To state a refinement of Markov's inequality, we need one more notation. Let $|\mathbf{x}| \leqslant 1$, let $\mathbf{v}$ be a unit vector, and l the line through $\mathbf{x}$ in the direction $\mathbf{v}$, and consider, if l does not pass through the origin, the intersection of $B(0, 1)$ with the two-dimensional subspace of $\mathbf{R}^k$ which contains $\mathbf{x}$ and $\mathbf{v}$. We denote by $D(l)$ the part of this intersection which lies between l and $-l$, where $-l$ is the line consisting of the points $-\mathbf{p}$, $\mathbf{p} \in l$. If l passes through the origin, $D(l)$ denotes the diameter in the direction $\mathbf{v}$.

Theorem 2′. *Let $P \in P_n$, $|\mathbf{x}| \leqslant 1$, and let $\mathbf{v}$ be a unit vector. Then*

$$|P_{\mathbf{v}}(\mathbf{x})| \leqslant n^2 \|P\|_{D(l)}.$$

Again we give the proofs in two dimensions, which is enough. We prepare for the proofs by giving a lemma, which is a version of Bernstein's inequality.

Lemma 2. *Let $S(0, 1)$ be the unit sphere in $\mathbf{R}^3$, let $\mathbf{t}$ be a unit tangent vector at a point $\mathbf{x}$ on the sphere, and let P be a polynomial in three variables of degree $\leqslant n$. Then $|P_{\mathbf{t}}(\mathbf{x})| \leqslant n \|P\|_G$, where G is the great circle through $\mathbf{x}$ which has $\mathbf{t}$ as a tangent vector.*

Proof. We may assume that G is the unit circle in the xy-plane (otherwise rotate the coordinate system). Let $T(\theta) = P(\cos\theta, \sin\theta, 0)$. Then, for θ corresponding to $\mathbf{x}$, by Bernstein's inequality for trigonometric polynomials, $|P_{\mathbf{t}}(\mathbf{x})| = |T'(\theta)| \leqslant n \|T\| = n \|P\|_G$.

Proof of Theorem 1′. In this proof we want to consider $\mathbf{x}$ in the theorem as a fixed point and hence we denote it by $\mathbf{x}_0 = (x_0, y_0)$. Furthermore, we assume, as we may, that $\mathbf{v}$ is parallel to the y-axis. Consider P as a polynomial in three variables by setting $P(x, y, z) = P(x, y)$. Take $z_0 > 0$ so that $\mathbf{x}_0' = (x_0, y_0, z_0)$ is a point on $S(0, 1)$. Let D be the circle obtained by intersecting $S(0, 1)$ with the plane $x = x_0$, let $\mathbf{t}$ be the unit tangent vector $(0, -z_0/d, y_0/d)$ to this circle at $\mathbf{x}_0'$, and let finally G be the great circle through $\mathbf{x}_0'$ which has $\mathbf{t}$ as a tangent. Denote by $E(\mathbf{x}_0)$ the projection of G onto the xy-plane; note that $E(\mathbf{x}_0)$ is the ellipse centered at the origin with major axis of length 2 which touches L at $\mathbf{x}_0$.

By Lemma 2 we have $|P_{\mathbf{t}}(\mathbf{x}_0')| \leqslant n \,\|P\|_G = n \,\|P\|_{E(\mathbf{x}_0)}$. But we also have $P_{\mathbf{t}}(\mathbf{x}_0') = -(z_0/d)(\partial P/\partial y)(\mathbf{x}_0') + (y_0/d)(\partial P/\partial z)(\mathbf{x}_0')$, so, since P does not depend on z, $z_0 = \sqrt{d^2 - y_0^2}$, and $d^2 - y_0^2 = 1 - |\mathbf{x}_0|^2$ by the theorem of intersecting chords, we get

$$|P_{\mathbf{v}}(\mathbf{x}_0)| \leqslant \frac{nd}{\sqrt{1-|\mathbf{x}_0|^2}} \,\|P\|_{E(\mathbf{x}_0)} \tag{2}$$

and the theorem is proved.

Proof of Theorem 2'. If $|\mathbf{x}| = 1$ and $\mathbf{v}$ is a tangent vector to the circle, the result follows from Lemma 2. Otherwise, let again $\mathbf{p}$ be the midpoint of the chord L through $\mathbf{x}$ in the direction $\mathbf{v}$, and put $Q(t) = P_{\mathbf{v}}(\mathbf{w})$ where $\mathbf{w} = \mathbf{p} + t\mathbf{v}$, $t \in \mathbf{R}$. Then for $|t| < d$ we have $E(\mathbf{w}) \subset D(l)$ and, by the theorem of intersecting chords, $1 - |\mathbf{w}|^2 = d^2 - t^2$. From (2) it follows that $|Q(t)| \leqslant nd/\sqrt{d^2 - t^2} \,\|P\|_{D(l)}$, $|t| < d$, and by means of Lemma 1 we get $|Q(t)| \leqslant (n/d)\, nd \,\|P\|_{D(l)} = n^2 \,\|P\|_{D(l)}$, $|t| \leqslant d$, which proves the theorem.

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